

# THE CLASSICAL DYNAMIC SYMMETRY FOR THE $\mathrm{Sp}(1)$ -KEPLER PROBLEMS

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**ABSTRACT.** A Poisson realization of the simple real Lie algebra  $\mathfrak{so}^*(4n)$  on the phase space of each  $\mathrm{Sp}(1)$ -Kepler problem is exhibited. As a consequence one obtains the Laplace-Runge-Lenz vector for each classical  $\mathrm{Sp}(1)$ -Kepler problem. The verification of these Poisson realizations is greatly simplified via an idea due to A. Weinstein. The totality of these Poisson realizations is shown to be equivalent to the canonical Poisson realization of  $\mathfrak{so}^*(4n)$  on the Poisson manifold  $T^*\mathbb{H}_*^n/\mathrm{Sp}(1)$ . (Here  $\mathbb{H}_*^n := \mathbb{H}^n \setminus \{0\}$  and the Hamiltonian action of  $\mathrm{Sp}(1)$  on  $T^*\mathbb{H}_*^n$  is induced from the natural right action of  $\mathrm{Sp}(1)$  on  $\mathbb{H}_*^n$ .)

**Keywords.** Kepler problem, Jordan algebra, dynamic symmetry, Laplace-Runge-Lenz vector, Weinstein's universal phase space.

## 1. INTRODUCTION

The Kepler problem is a textbook example of super integrable models. Its hamiltonian is invariant under the Lie group  $\mathrm{SO}(4)$ , larger than the manifest symmetry group  $\mathrm{SO}(3)$ . A remarkable fact about the Kepler problem is that the real non-compact Lie algebra  $\mathfrak{so}(4, 2)$  has a nontrivial Poisson realization on its phase space. This Poisson realization, more precisely its quantized form, was initially discovered by I. A. Malkin and V. I. Manko [1] in 1966. (For the prehistory of this important discovery about the Kepler problem, one may consult Footnote 2 in Ref. [2].) In the literature, the real non-compact Lie algebra  $\mathfrak{so}(4, 2)$  is referred to as the *dynamical symmetry algebra* for the Kepler problem and its afore-mentioned Poisson realization is referred to as the *classical dynamical symmetry* for the Kepler problem.

The Kepler problem has magnetized versions, under the name of MICZ-Kepler problems. The work of A. Barut and G. Bornzin [3], extends the study of dynamical symmetry to these magnetized Kepler problems at the quantum level. Later, the higher dimensional analogue of MICZ-Kepler problems, under the name of generalized MICZ-Kepler problems, were found [4] and the dynamical symmetry for these models was studied as well, at both the quantum level [5] and the classical level [6].

About six years ago the second author [7, 8] discovered that the Kepler problem has a vast generalization based on simple euclidean Jordan algebra, for which the conformal algebra of the Jordan algebra is the dynamical symmetry algebra. He also made the following observation [9]: for a generalized Kepler problem, its hamiltonian and its Laplace-Runge-Lenz vector can all be derived from its dynamical symmetry.

Recently, we exhibited in Ref. [10] the classical dynamical symmetry for the  $\mathrm{U}(1)$ -Kepler problems (see Ref. [11]). These Kepler-type problems are naturally associated with the euclidean Jordan algebras of complex hermitian matrices. In this article we shall

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exhibit the classical dynamical symmetry for the  $\mathrm{Sp}(1)$ -Kepler problems [12], i.e. the quaternionic analogues of the  $\mathrm{U}(1)$ -Kepler problems. As a result, we obtain the Laplace-Runge-Lenz vector for each  $\mathrm{Sp}(1)$ -Kepler problem as well as a formulae of expressing the total energy in terms of the angular momentum and the Laplace-Runge-Lenz vector. For the convenience of readers, we end this introduction with

### A list of symbols

$\mathbb{H}$	the set of quaternions
$H_n(\mathbb{H})$	the Jordan algebra of quaternionic hermitian matrices of order $n$
$Im\ \mathbb{H}$	the set of imaginary quaternions
$i, j, k$	the standard orthonormal basis for $Im\ \mathbb{H}$ such that $ij = k$
$\bar{q}$	the quaternionic conjugate of quaternion $q$ , e.g., $\bar{k} = -k$
$\Re q$	$\frac{1}{2}(q + \bar{q})$
$Im\ q$	$\frac{1}{2}(q - \bar{q})$
$\lrcorner$	the interior product of vectors with forms
$\wedge$	the wedge product of forms
$d$	the exterior derivative operator
$\pi_X : T^*X \rightarrow X$	the cotangent bundle projection
$G$	a compact connected Lie group
$\mathfrak{g}, \mathfrak{g}^*$	the Lie algebra of $G$ and its dual
$\xi$	an element in $\mathfrak{g}$
$\langle \cdot, \cdot \rangle$	either the pairing of vectors with co-vectors or inner product
$\langle   \rangle$	the inner product on the Jordan algebra $H_n(\mathbb{H})$
$\mathrm{Ad}_a$	the adjoint action of $a \in G$ on $\mathfrak{g}$
$P \rightarrow X$	a principal $G$ -bundle
$\Theta$	a $\mathfrak{g}$ -valued differential one-form on $P$ that defines a principal connection on $P \rightarrow X$
$R_a$	the right action on $P$ by $a \in G$
$X_\xi$	the vector field on $P$ which represents the infinitesimal right action on $P$ by $\xi \in \mathfrak{g}$
$F$	a hamiltonian $G$ -space
$\mathcal{F} := P \times_G F$	the quotient of $P \times F$ by the action of $G$
$\Phi : F \rightarrow \mathfrak{g}^*$	the $G$ -equivariant moment map
$\mathcal{F}^\#$	Sternberg phase space
$\mathcal{W}$	Weinstein's universal phase space

## 2. THE DYNAMICAL SYMMETRY ALGEBRA

The dynamic symmetry for the  $\mathrm{Sp}(1)$ -Kepler model at level  $n$  with magnetic charge  $\mu$  that we shall exhibit is a Poisson realization of the dynamic symmetry algebra  $\mathfrak{so}^*(4n)$  on its phase space. Note that  $\mathfrak{so}^*(4n)$ , being the conformal algebra of the simple euclidean Jordan algebra  $H_n(\mathbb{H})$  of quaternionic hermitian matrices of order  $n$ , can be understood naturally in the language of Jordan algebra [13]. The details are given in the next two paragraphs (see [14] for more details).

For each  $u \in V := H_n(\mathbb{H})$ , we use  $L_u$  to denote the Jordan multiplication by  $u$ , and for each  $u, v \in V$ , we let  $S_{uv} = [L_u, L_v] + L_{uv}$  where  $[L_u, L_v]$  stands for the commutator:  $L_u L_v - L_v L_u$  and  $uv$  in  $L_{uv}$  means  $L_u(v)$ , i.e., the symmetrized matrix product of  $u$  with  $v$ . We use  $\{uvw\}$  to denote  $S_{uv}(w)$ . Then we have

$$[S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}} \quad \text{for any } u, v, z, w \text{ in } V.$$

So these  $S_{uv}$  span a real Lie algebra. This Lie algebra is denoted by  $\mathfrak{str}$ , and is referred to as the structure algebra of  $V$ . In fact  $\mathfrak{str} = \mathfrak{su}^*(2n) \oplus \mathbb{R}$  where the center  $\mathbb{R}$  is generated by  $L_e$  — the Jordan multiplication by the Jordan identity element  $e$ .

The conformal algebra  $\mathfrak{co}$  is an extension of the structure algebra  $\mathfrak{str}$ . As a real vector space we have

$$\mathfrak{co} = V \oplus \mathfrak{str} \oplus V^*.$$

An element  $z$  in  $V$ , rewritten as  $X_z$ , behaves like a vector:

$$[S_{uv}, X_z] = X_{\{uvz\}}$$

and an element in  $V^*$  behaves like a co-vector. Via the inner product on  $V$ , we can identify this element in  $V^*$  with an element  $w$  in  $V$ , which is rewritten as  $Y_w$ , then

$$[S_{uv}, Y_w] = -Y_{\{vuw\}}.$$

The remaining commutation relations are

$$[X_u, Y_v] = 0, \quad [Y_u, Y_v] = 0, \quad [X_u, X_v] = -2S_{uv} \quad \text{for any } u, v \text{ in } V.$$

One can verify that, indeed,  $\mathfrak{co} = \mathfrak{so}^*(4n)$ .

### 3. THE PHASE SPACE

When the magnetic charge is not zero, the phase space, being a Sternberg phase space [15], is a bit involved, so the Poisson realization of the dynamic symmetry algebra on the the phase space is a bit complicated and the verification of various Poisson relations becomes quite tedious, as evidenced already in simpler models such as the  $\mathrm{U}(1)$ -Kepler problems [10].

To circumvent this complication, we resort to an insight of A. Weinstein into the Sternberg phase space. As we shall see the Sternberg phase spaces form a bundle of symplectic manifolds over the affine space of principal connections. Since its base space is contractible, this fiber bundle must be topologically trivial. Indeed, A. Weinstein observed that [16] this bundle of symplectic manifolds is canonically isomorphic to a product bundle whose fiber is a fixed symplectic manifold, i.e., Weinstein's *universal phase space*.

**3.1. Review of the work by S. Sternberg and A. Weinstein.** The goal of this subsection is to review the work by S. Sternberg [15] and A. Weinstein [16]. Let us start with the setup for Sternberg phase space:

- (i) A compact Lie group  $G$  and a principal  $G$ -bundle  $P \rightarrow X$  with a principal connection form  $\Theta$ ,
- (ii) A Hamiltonian  $G$ -space  $F$  with symplectic form  $\Omega$  and a  $G$ -equivariant moment map  $\Phi: F \rightarrow \mathfrak{g}^*$ . Here  $\mathfrak{g}$  is the Lie algebra of the Lie group  $G$ .

Note that, a co-adjoint orbit of  $G$  with the Kirillov-Kostant-Souriau symplectic form is a typical example of Hamiltonian  $G$ -space.

For the convenience of readers, let us recall that a principal connection  $\Theta$  on the principal  $G$ -bundle  $P \rightarrow X$  is a  $\mathfrak{g}$ -valued differential one-form on  $P$  satisfying the following two conditions:

$$1) R_{a^{-1}}^* \Theta = \mathrm{Ad}_a \Theta \text{ for any } a \in G, \quad 2) \Theta(X_\xi) = \xi \text{ for any } \xi \in \mathfrak{g}.$$

Here  $a \in G$ ,  $R_{a^{-1}}$  is the right multiplication of  $a^{-1}$  on  $P$ ,  $\text{Ad}_a$  is the adjoint action of  $a$  on  $\mathfrak{g}$ , and  $X_\xi$  is the vector field on  $P$  which represents the induced action of  $\xi \in \mathfrak{g}$  on  $P$ , i.e., for any  $f \in C^\infty(P)$ , we have the Lie derivative

$$(3.1) \quad \mathcal{L}_{X_\xi} f \Big|_p = \frac{d}{dt} \Big|_{t=0} f(p \cdot \exp(t\xi)).$$

It is easy to see that  $[\mathcal{L}_{X_\xi}, \mathcal{L}_{X_\eta}] = \mathcal{L}_{X_{[\xi, \eta]}}$ , or equivalently  $[X_\xi, X_\eta] = X_{[\xi, \eta]}$ .

It is a tautology that a smooth (right)  $G$ -action on a smooth manifold  $P$  yields a hamiltonian  $G$ -action on the symplectic manifold  $T^*P$  with the  $G$ -equivariant moment map  $\rho: T^*P \rightarrow \mathfrak{g}^*$ . Indeed, if we use  $\pi_P$  to denote the bundle projection  $T^*P \rightarrow P$ , then  $\rho$  is defined via equation

$$\langle \rho(z), \xi \rangle = \langle z, X_\xi(\pi_P(z)) \rangle.$$

Another way to see it is this: for each point  $p \in P$ , the dual of linear map

$$(3.2) \quad \begin{array}{ccc} \mathfrak{g} & \rightarrow & T_p P \\ \xi & \mapsto & X_\xi(p) \end{array}$$

is a linear map from  $T_p^*P$  to  $\mathfrak{g}^*$ . Assembling these dual maps together, we get the map  $\rho: T^*P \rightarrow \mathfrak{g}^*$ .

The  $G$ -equivariance of  $\rho$  is reflected by the fact that  $(R_g)_*(X_\xi) = X_{\text{Ad}_{g^{-1}}(\xi)}$  and the fact that  $\text{Ad}_g^*(\alpha) = \alpha \circ \text{Ad}_{g^{-1}}$  for any  $\alpha \in \mathfrak{g}^*$ . The map  $\rho$  is a moment map, i.e.,  $d\langle \rho, \xi \rangle = -\hat{X}_\xi \lrcorner \omega_P$ . Here  $\omega_P$  is the canonical symplectic form on  $T^*P$ , and the vector field  $\hat{X}_\xi$  on  $T^*P$  is the cotangent lift of the vector field  $X_\xi$ . In local coordinates, we have

$$z = p_i(z) dx^i|_{\pi_P(z)}, \quad \omega_P = dp_i \wedge dx^i, \quad X_\xi = X_\xi^i \frac{\partial}{\partial x^i}, \quad \hat{X}_\xi = X_\xi^i \frac{\partial}{\partial x^i} - p_i \frac{\partial X_\xi^i}{\partial x^j} \frac{\partial}{\partial p_j}$$

and  $\langle \rho, \xi \rangle = p_i X_\xi^i$ . Here  $x^i$  is a system of local coordinates on  $P$ .

Combining  $\rho$  and  $\Phi$ , we obtain a  $G$ -equivariant moment map

$$\psi: T^*P \times F \rightarrow \mathfrak{g}^*.$$

which maps  $(x, y)$  to  $-\rho(x) + \Phi(y)$ .

Since  $P \rightarrow X$  is a principal  $G$ -bundle,  $\rho|_{T_p^*P}: T_p^*P \rightarrow \mathfrak{g}^*$  is a diffeomorphism for each  $p \in P$ , then  $\rho: T^*P \rightarrow \mathfrak{g}^*$  is a submersion. Consequently  $\psi$  is a submersion as well. In particular, this means that  $\psi^{-1}(0)$  is a submanifold of  $T^*P \times F$ . Since the isotropic group of  $0 \in \mathfrak{g}^*$ , being the Lie group  $G$ , is compact, its free action on  $\psi^{-1}(0)$  is proper, Theorem 1 in Ref. [17] applies, so there is a unique symplectic structure  $\omega$  on  $\psi^{-1}(0)/G$  such that  $\pi^*\omega = \iota^*(\omega_P + \Omega)$  where  $\omega_P$  is the tautological symplectic form on  $P$ ,  $\pi$  is the projection and  $\iota$  is the inclusion:

$$\begin{array}{ccc} \psi^{-1}(0) & \xrightarrow{\iota} & T^*P \times F \\ \pi \downarrow & & \\ \psi^{-1}(0)/G & & \end{array}$$

In the literature this reduced phase space  $(\psi^{-1}(0)/G, \omega)$  is called the Weinstein's *universal phase space*  $\mathcal{W}$ . Note that, no connection is required for the existence of this universal phase space.

To understand the meaning of the word “universal”, let us suppose that a principal connection  $\Theta$  on  $P \rightarrow X$  is given. For each point  $p \in P$ , let  $x$  be the image of  $p$  under the bundle projection map, then the equivariant horizontal lifting of tangent vectors on  $X$

(provided by  $\Theta$ ) defines a linear map  $T_x X \rightarrow T_p P$ . By assembling the dual of these linear maps together, we arrive at the commutative square

$$\begin{array}{ccc} T^*P & \longrightarrow & T^*X \\ \pi_P \downarrow & & \downarrow \pi_X \\ P & \longrightarrow & X \end{array}$$

where the top arrow, fiber-wise speaking, is the dual of the horizontal lifting of tangent vectors of  $X$  to tangent vectors of  $P$ . Let  $\tilde{P}$  be the pullback of

$$\begin{array}{ccc} & T^*X & \\ & \downarrow \pi_X & \\ P & \longrightarrow & X \end{array}$$

then we have a  $G$ -equivariant map  $T^*P \rightarrow \tilde{P}$ , hence, by taking its product with the identity map on  $F$ , we obtain a  $G$ -equivariant map

$$\alpha_\Theta : T^*P \times F \rightarrow \tilde{P} \times F.$$

Next, Weinstein observed that the restriction of  $\alpha_\Theta$  to  $\psi^{-1}(0)$  is a diffeomorphism; then, passing to the quotient by the action of  $G$ , one obtains a diffeomorphism

$$\bar{\alpha}_\Theta : \psi^{-1}(0)/G \rightarrow \tilde{P} \times_G F.$$

So there is a unique symplectic structure  $\omega_\Theta$  on  $\mathcal{F}^\sharp := \tilde{P} \times_G F$  such that  $\bar{\alpha}_\Theta^*(\omega_\Theta) = \omega$ . Then  $(\mathcal{F}^\sharp, \omega_\Theta)$  is a symplectic manifold, which is precisely the Sternberg phase space of  $\Theta$  described in Ref. [15]. In other words, the Sternberg phase spaces form a bundle of symplectic manifolds over the space of principal connections, and this bundle is canonically isomorphic to the product bundle whose fiber is Weinstein's universal phase space  $\mathcal{W}$ .

**3.2. Sternberg phase space for  $\mathrm{Sp}(1)$ -Kepler problems.** The classical  $\mathrm{Sp}(1)$ -Kepler problems (or models) are indexed by integer (called *level*)  $n \geq 2$  and real number (called *magnetic charge*)  $\mu \geq 0$ . For the model at level  $n$  and magnetic charge  $\mu$ , its phase space is the Sternberg phase space  $\mathcal{F}_\mu^\sharp$  with the following data [12]:

- (i) The compact Lie group  $G$  is  $\mathrm{Sp}(1)$  (i.e.  $\mathrm{SU}(2)$ ) and the principal  $G$ -bundle is

$$(3.3) \quad \begin{array}{ccc} \mathbb{H}_*^n & \rightarrow & \mathcal{C}_1 \\ Z & \mapsto & nZZ^\dagger \end{array}$$

and the principal connection form is

$$(3.4) \quad \Theta = \frac{\mathrm{Im}(\bar{Z} \cdot dZ)}{|Z|^2}.$$

Here  $\mathcal{C}_1$  is the rank-one Kepler cone for the simple Euclidean Jordan algebra  $\mathrm{H}_n(\mathbb{H})$  of quaternionic hermitian matrices of order  $n$ . As a submanifold of the Euclidean space  $\mathrm{H}_n(\mathbb{H})$ ,  $\mathcal{C}_1$  consists of all rank one semi-positive definite elements of  $\mathrm{H}_n(\mathbb{H})$ . However,  $\mathcal{C}_1$  is not a Riemannian submanifold of the Euclidean space  $\mathrm{H}_n(\mathbb{H})$  because the Riemannian metric on  $\mathcal{C}_1$ , called the Kepler metric, does not come from the Euclidean metric via restriction, see Ref. [12] for the details.

- (ii) For simplicity, we shall identify  $\mathfrak{g}^*$  with  $\mathfrak{g} := \text{Im } \mathbb{H}$  via the standard invariant inner product  $\langle \cdot, \cdot \rangle$ , i.e. the one such that the imaginary units  $i, j$  and  $k$  form an orthonormal basis for  $\mathfrak{g}$ . Then the Hamiltonian  $G$ -space is

$$(3.5) \quad F := \{\xi \in \text{Im } \mathbb{H} \mid \xi \bar{\xi} = \mu^2\}$$

whose symplectic form  $\Omega_\mu$ , being the Kirillov-Kostant-Souriau symplectic form, is given by the formulae

$$(3.6) \quad \Omega_\mu = \frac{\langle \xi, d\xi \wedge d\xi \rangle}{2|\xi|^2}.$$

Note that, if we write  $\xi = \xi^1 i + \xi^2 j + \xi^3 k$ , then this symplectic form yields the following basic Poisson relations on  $F$ :

$$(3.7) \quad \{\xi^1, \xi^2\} = \xi^3, \quad \{\xi^2, \xi^3\} = \xi^1, \quad \{\xi^3, \xi^1\} = \xi^2.$$

(Note: In our convention,  $ij = k$ .)

- (iii) The  $G$ -equivariant moment map is

$$(3.8) \quad \begin{array}{ccc} \Phi : F & \rightarrow & \text{Im } \mathbb{H} \\ \xi & \mapsto & 2\xi \end{array}$$

(Here the  $G$ -action on both  $F$  and  $\text{Im } \mathbb{H}$  is the adjoint action.) Indeed,  $\Phi$  is (obviously)  $G$ -equivariant; moreover, with a seemingly odd factor of 2 included in Eq. (3.8), one can verify that

$$d\langle \Phi, \eta \rangle = X_\eta \lrcorner \Omega_\mu.$$

Here  $X_\eta$  is the vector field on  $F$  that represents the adjoint action on  $F$  by the Lie algebra element  $\eta$ , so  $X_\eta(\xi) = (\xi, [\eta, \xi]) \in T_\xi F$ .

**3.3. Weinstein's Universal Phase Space for  $\text{Sp}(1)$ -Kepler problems.** The right action of  $\text{Sp}(1)$  on  $\mathbb{H}_*^n$  is the map that sends  $(Z, q) \in \mathbb{H}_*^n \times \text{Sp}(1)$  to  $Z[q]$  — the matrix multiplication of the column matrix  $Z$  with the  $1 \times 1$ -matrix  $[q]$ . The action induces a hamiltonian  $\text{Sp}(1)$ -action on  $T^*\mathbb{H}_*^n$  with a tautological moment map.

The canonical trivialization of  $T\mathbb{H}_*^n$  yields two  $\mathbb{H}$ -valued function on  $T\mathbb{H}_*^n$ , i.e. the position vector  $Z$  and velocity vector  $W$ . Via the standard inner product on  $\mathbb{H}^n$ :

$$\langle U, V \rangle = \mathcal{R}\ell(U^\dagger V),$$

one can identify the total cotangent space  $T^*\mathbb{H}_*^n$  with the total tangent space  $T\mathbb{H}_*^n$ , then  $T\mathbb{H}_*^n$  become a Poisson manifold with the following basic Poisson relation: for any  $U, V \in \mathbb{H}^n$ ,

$$\{\langle U, Z \rangle, \langle V, W \rangle\} = \langle U, V \rangle, \quad \{\langle U, Z \rangle, \langle V, Z \rangle\} = \{\langle U, W \rangle, \langle V, W \rangle\} = 0.$$

With the aforementioned identification of  $T^*\mathbb{H}_*^n$  with  $T\mathbb{H}_*^n$  and  $\mathfrak{g}^*$  with  $\mathfrak{g}$  as well, one can check that the moment map  $\rho$  is identified with the map from  $T\mathbb{H}_*^n$  to  $\mathfrak{g}$  that maps  $(Z, W)$  to  $-\text{Im}(W^\dagger Z)$ . Therefore, the moment map  $\psi: T\mathbb{H}_*^n \times F \rightarrow \mathfrak{g}$  is

$$(3.9) \quad \psi(Z, W, \xi) = \text{Im}(W^\dagger Z) + 2\xi.$$

Note that the action of  $g$  maps  $(Z, W, \xi)$  to  $(Z \cdot g^{-1}, W \cdot g^{-1}, g\xi g^{-1})$ .

The map  $\psi$  has a natural extension to  $\tilde{\psi}: T\mathbb{H}_*^n \times \mathfrak{g} \rightarrow \mathfrak{g}$  which is defined by the same formulae:  $\tilde{\psi}(Z, W, \xi) = \text{Im}(W^\dagger Z) + 2\xi$ . Then  $\tilde{\psi}^{-1}(0)$  is the graph of the map  $\xi = -\frac{1}{2}\text{Im}(W^\dagger Z)$ . This map is clearly  $G$ -equivariant. Moreover, it is a Poisson map. Indeed, for example,

$$\{\langle i, W^\dagger Z \rangle, \langle j, W^\dagger Z \rangle\} = \{\langle Wi, Z \rangle, \langle Wj, Z \rangle\}$$

$$\begin{aligned}
&= \{ \langle Wi, Z \rangle, \overline{\langle Wj, Z \rangle} \} - \langle i \leftrightarrow j \rangle \\
&= -\{ \langle Wi, Z \rangle, \langle W, Zj \rangle \} - \langle i \leftrightarrow j \rangle \\
&= -\langle Wi, Zj \rangle - \langle i \leftrightarrow j \rangle \\
&= -2\langle k, W^\dagger Z \rangle.
\end{aligned}$$

So, in view of the fact that  $\xi^1 = -\frac{1}{2}\langle i, W^\dagger Z \rangle$ ,  $\xi^2 = -\frac{1}{2}\langle j, W^\dagger Z \rangle$ ,  $\xi^3 = -\frac{1}{2}\langle k, W^\dagger Z \rangle$ , we have  $\{\xi^1, \xi^2\} = \xi^3$ , cf. Eq. (3.7).

Therefore, the  $\mathrm{Sp}(1)$ -equivariant projection map  $T\mathbb{H}_*^n \times \mathfrak{g} \rightarrow T\mathbb{H}_*^n$ , when restricted to  $\tilde{\psi}^{-1}(0)$ , yields a  $\mathrm{Sp}(1)$ -equivariant Poisson isomorphism of  $\tilde{\psi}^{-1}(0)$  with  $T\mathbb{H}_*^n$ . Since  $\psi^{-1}(0)$  is a  $\mathrm{Sp}(1)$ -equivariant submanifold of  $\tilde{\psi}^{-1}(0)$ , the universal phase space

$$\mathcal{W}_\mu := \psi^{-1}(0)/\mathrm{Sp}(1) = \left\{ \mathrm{Sp}(1) \cdot (Z, W, \xi) \mid \xi = -\frac{1}{2} \mathrm{Im}(W^\dagger Z), |\xi| = \mu \right\}$$

is naturally identified with a submanifold of  $T\mathbb{H}_*^n/\mathrm{Sp}(1)$ . Indeed, as a symplectic manifold,  $\mathcal{W}_\mu$  is naturally identified with the Poisson leave

$$\{ \mathrm{Sp}(1) \cdot (Z, W) \mid |\mathrm{Im}(W^\dagger Z)| = 2\mu \}$$

of the Poisson manifold  $T\mathbb{H}_*^n/\mathrm{Sp}(1)$ . In other words, in view of the fact that the Sternberg phase space  $\mathcal{F}_\mu^\sharp$  can be identified with the Weinstein's universal phase space  $\mathcal{W}_\mu$ , *the totality of Sternberg phase space can be identified with the Poisson manifold  $T\mathbb{H}_*^n/\mathrm{Sp}(1)$ .*

Part (ii) of Lemma 1 in the next section says that *the conformal algebra of  $\mathrm{H}_n(\mathbb{H})$  has a Poisson realization on the Poisson manifold  $T\mathbb{H}_*^n/\mathrm{Sp}(1)$* , hence on both the universal phase space  $\mathcal{W}_\mu$  and the Sternberg phase space  $\mathcal{F}_\mu^\sharp$ .

#### 4. DYNAMICAL SYMMETRY

Let us fix a  $\mathrm{Sp}(1)$ -Kepler problem, say at level  $n$  and with magnetic charge  $\mu$ . Its phase space, being the Sternberg phase space  $\mathcal{F}_\mu^\sharp$ , fibers over  $T^*\mathcal{C}_1$ . Via the metric on the Euclidean space  $V := \mathrm{H}_n(\mathbb{H})$ ,  $T^*\mathcal{C}_1$  can be identified with  $TC_1$ , so  $\mathcal{F}_\mu^\sharp$  fibers over  $TC_1$  as well. Let functions  $x$  and  $\pi$  (taking value in vector space  $V$ ) be defined via diagram

$$\begin{array}{ccc}
& & TC_1 \\
& \swarrow & \downarrow \iota \\
& TV[\text{description}] & \\
& \swarrow \tau_V & \searrow \\
V & & V
\end{array}$$

where  $\iota$  is the inclusion map and  $\tau_V$  is the tangent bundle projection and  $t$  is the natural trivialization map of the tangent bundle of the affine space  $V$ . The pullback of  $x$  and  $\pi$  under the bundle map  $\mathcal{F}_\mu^\sharp \rightarrow TC_1$  shall still be denoted by  $x$  and  $\pi$ . Recall that the inner product on the Jordan algebra  $\mathrm{H}_n(\mathbb{H})$  is denoted by  $\langle \mid \rangle$ , so if  $u \in \mathrm{H}_n(\mathbb{H})$ , then  $\langle x \mid u \rangle$  is a real function on  $\mathcal{F}_\mu^\sharp$ .

**Theorem 1.** *For the conformal algebra of the Jordan algebra  $\mathrm{H}_n(\mathbb{H})$ , there is a unique Poisson realization on the Sternberg phase space  $\mathcal{F}_\mu^\sharp$  such that  $Y_u$  is realized as function  $\mathcal{Y}_u := \langle x \mid u \rangle$  and  $X_e$  is realized as function  $\mathcal{X}_e := \langle x \mid \pi^2 \rangle + \frac{\mu^2}{\langle e \mid x \rangle}$ . Moreover, if  $(e_\alpha)$*

is an orthonormal basis for  $H_n(\mathbb{H})$  and  $\mathcal{L}_u$  represents  $L_u$  in this Poisson realization, we have the following primary quadratic relation:

$$(4.1) \quad \frac{2}{n} \sum_{\alpha} \mathcal{L}_{e_{\alpha}}^2 = \mathcal{L}_e^2 + \mathcal{X}_e \mathcal{Y}_e - \mu^2.$$

The uniqueness of this Poisson realization (if it exists) is clear, that is because  $X_e$  and  $Y_u$  generate the conformal algebra.

As for the existence of this Poisson realization, the direct verification is very complicated. Since Sternberg phase space can be identified with Weinstein's universal phase space via diffeomorphism

$$(4.2) \quad \bar{\alpha}_{\Theta} : \mathcal{W}_{\mu} := \psi^{-1}(0)/\mathrm{Sp}(1) \rightarrow \mathcal{F}_{\mu}^{\sharp} := \tilde{P} \times_{\mathrm{Sp}(1)} F,$$

one just needs to verify the existence of the corresponding Poisson realization on Weinstein's universal phase space  $\mathcal{W}_{\mu}$ , a task which turns to be much simpler.

**Lemma 1.** (i). Let  $\mathrm{Sp}(1) \cdot (Z, W, \xi)$  be an element of  $\mathcal{W}_{\mu}$ , and let  $\mathcal{Y}_u$  and  $\mathcal{X}_e$  be the functions defined in Theorem 1. Then

$$\mathcal{Y}_u \circ \bar{\alpha}_{\Theta}(\mathrm{Sp}(1) \cdot (Z, W, \xi)) = \langle Z, uZ \rangle, \quad \mathcal{X}_e \circ \bar{\alpha}_{\Theta}(\mathrm{Sp}(1) \cdot (Z, W, \xi)) = \frac{1}{4}|W|^2.$$

(ii). For any vectors  $u, v$  in  $V := H_n(\mathbb{H})$ , define  $\mathrm{Sp}(1)$ -invariant functions

$$(4.3) \quad \begin{cases} \mathcal{X}_u &:= \frac{1}{4}\langle W, uW \rangle \\ \mathcal{Y}_v &:= \langle Z, vZ \rangle \\ \mathcal{S}_{uv} &:= \frac{1}{2}\langle W, (u \cdot v)Z \rangle \end{cases}$$

on  $T\mathbb{H}_*^n$ . Here  $u \cdot v$  means the matrix multiplication of  $u$  with  $v$ . Then, for any vectors  $u, v, z, w$  in  $V$ , the following Poisson bracket relations hold:

$$\begin{cases} \{\mathcal{X}_u, \mathcal{X}_v\} = 0, & \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, & \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}, \\ \{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, & \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}, \\ \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_z\{vuw\}. \end{cases}$$

Consequently, we have a Poisson realization on the Poisson manifold  $T^*\mathbb{H}_*^n/\mathrm{Sp}(1)$  for the conformal algebra of the Jordan algebra  $H_n(\mathbb{H})$ .

(iii). Let  $e$  be the identity element of  $V$ ,  $(e_{\alpha})$  be an orthonormal basis for  $V$ , and  $\mathcal{L}_u = \mathcal{S}_{eu}$  for any  $u \in V$ . Then

$$(4.4) \quad \frac{2}{n} \sum_{\alpha} \mathcal{L}_{e_{\alpha}}^2 = \mathcal{L}_e^2 + \mathcal{X}_e \mathcal{Y}_e - \mu^2.$$

*Proof.* (i). To understand the map  $\bar{\alpha}_{\Theta}$ , we need to figure out the horizontal lift induced from the connection  $\Theta$ . For  $Z \in \mathbb{H}_*^n$ , we let  $x = nZZ^{\dagger}$ . Suppose that  $(x, \dot{x})$  is a tangent vector of  $\mathcal{C}_1$  at  $x$ , and  $(Z, \dot{Z})$  is its horizontal lift to point  $Z$  in  $\mathbb{H}_*^n$ . Then, in view of Eq. (3.4), we have equations

$$n(\dot{Z}Z^{\dagger} + Z\dot{Z}^{\dagger}) = \dot{x}, \quad \mathrm{Im}(Z^{\dagger}\dot{Z}) = 0.$$

By solving these equations jointly, we obtain

$$\dot{Z} = \frac{1}{n|Z|^2}(\dot{x}Z - \frac{\mathrm{tr} \dot{x}}{2}Z).$$



Suppose that the  $\Theta$ -induced map  $T^*\mathbb{H}_*^n \rightarrow T^*\mathcal{C}_1$  maps  $(Z, \langle W, \cdot \rangle)$  to  $(x, \langle \pi | \cdot \rangle)$ , then  $x = nZZ^\dagger$  and

$$(4.5) \quad \langle \pi | \dot{x} \rangle = \frac{1}{n|Z|^2} \left\langle W, \dot{x}Z - \frac{\text{tr } \dot{x}}{2} Z \right\rangle.$$

In particular, in view of the fact that  $(x, ux) \in T_x\mathcal{C}_1$ , we have

$$(4.6) \quad \begin{aligned} \langle \pi | ux \rangle &= \frac{1}{n|Z|^2} \left\langle W, (ux)Z - \frac{\text{tr}(ux)}{2} Z \right\rangle \quad ux \text{ is the Jordan product of } u \text{ with } x \\ &= \frac{1}{2|Z|^2} \langle W, uZZ^\dagger Z + ZZ^\dagger uZ - \mathcal{R}_\ell \text{tr}(uZZ^\dagger)Z \rangle \\ &= \frac{1}{2} \langle W, uZ \rangle. \end{aligned}$$

Then

$$(4.7) \quad \bar{\alpha}_\Theta(G \cdot (Z, W, \xi)) = G \cdot (Z, nZZ^\dagger, \pi, \xi).$$

Consequently

$$\mathcal{Y}_u \circ \bar{\alpha}_\Theta(G \cdot (Z, W, \xi)) = \langle x | u \rangle = \langle nZZ^\dagger | u \rangle = \mathcal{R}_\ell \text{tr}(ZZ^\dagger u) = \langle Z, uZ \rangle$$

and

$$\begin{aligned} \mathcal{X}_e \circ \bar{\alpha}_\Theta(G \cdot (Z, W, \xi)) &= \langle x | \pi^2 \rangle + \frac{\mu^2}{\langle e | x \rangle} = \langle \pi | \pi x \rangle + \frac{n\mu^2}{\text{tr } x} \\ &= \frac{1}{2} \langle W, \pi Z \rangle + \frac{\mu^2}{|Z|^2} \quad \text{using Eq. (4.6)} \\ &= \frac{n}{2} \langle \pi | (ZW^\dagger)_+ \rangle + \frac{\mu^2}{|Z|^2}. \end{aligned}$$

Here  $(ZW^\dagger)_+ = \frac{1}{2}(ZW^\dagger + WZ^\dagger)$ . One can check that  $(x, (ZW^\dagger)_+)$  is a tangent vector of  $\mathcal{C}_1$  at  $x$ . Therefore, in view of Eq. (4.5), we have

$$\begin{aligned} \mathcal{X}_e \circ \bar{\alpha}_\Theta(G \cdot (Z, W, \xi)) &= \frac{1}{2|Z|^2} \left\langle W, (ZW^\dagger)_+ Z - \frac{\text{tr}(ZW^\dagger)_+}{2} Z \right\rangle + \frac{\mu^2}{|Z|^2} \\ &= \frac{1}{2|Z|^2} \left\langle W, (ZW^\dagger)_+ Z - \frac{\langle W, Z \rangle}{2} Z \right\rangle + \frac{\mu^2}{|Z|^2} \\ &= \frac{1}{2|Z|^2} \langle W, (ZW^\dagger)_+ Z \rangle - \frac{1}{4|Z|^2} \langle W, Z \rangle^2 + \frac{\mu^2}{|Z|^2} \\ &= \frac{1}{4|Z|^2} (|W|^2|Z|^2 + \mathcal{R}_\ell(W^\dagger Z)^2 - \langle W, Z \rangle^2) + \frac{\mu^2}{|Z|^2} \\ &= \frac{1}{4|Z|^2} (|W|^2|Z|^2 + (\text{Im}(W^\dagger Z))^2) + \frac{\mu^2}{|Z|^2} \\ &= \frac{1}{4|Z|^2} (|W|^2|Z|^2 - |\text{Im}(W^\dagger Z)|^2) + \frac{\mu^2}{|Z|^2} \\ &= \frac{1}{4} |W|^2. \end{aligned}$$

(ii). It is clear that  $\{\mathcal{X}_u, \mathcal{X}_v\} = 0$  and  $\{\mathcal{Y}_u, \mathcal{Y}_v\} = 0$ . Next, we have

$$\begin{aligned} \{\mathcal{X}_u, \mathcal{Y}_v\} &= \frac{1}{4} \{ \langle W, uW \rangle, \langle Z, vZ \rangle \} \\ &= \left\{ \overline{\langle W, uW \rangle}, \langle Z, vZ \rangle \right\} = -\langle uW, vZ \rangle \\ &= -2\mathcal{S}_{uv} \end{aligned}$$

and

$$\begin{aligned}
\{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} &= \frac{1}{4} \{ \langle W, (u \cdot v)Z \rangle, \langle W, (z \cdot w)Z \rangle \} \\
&= \frac{1}{4} \left\{ \overline{\langle W, (u \cdot v)Z \rangle}, \overline{\langle W, (z \cdot w)Z \rangle} \right\} - \langle (u \cdot v) \leftrightarrow (z \cdot w) \rangle \\
&= -\frac{1}{4} \langle (z \cdot w)^\dagger W, (u \cdot v)Z \rangle + \frac{1}{4} \langle (u \cdot v)^\dagger W, (z \cdot w)Z \rangle \\
&= \frac{1}{4} \langle W, [u \cdot v, z \cdot w]Z \rangle \\
&= \mathcal{S}_{\{uvz\}w} - \mathcal{S}_z\{vuw\}
\end{aligned}$$

because  $\{uvz\} = \frac{1}{2}(u \cdot v \cdot z + z \cdot v \cdot u)$  and  $\{vuw\} = \frac{1}{2}(v \cdot u \cdot w + w \cdot u \cdot v)$ .

Thirdly,

$$\begin{aligned}
\{\mathcal{S}_{uv}, \mathcal{X}_z\} &= \frac{1}{8} \{ \langle W, (u \cdot v)Z \rangle, \langle W, zW \rangle \} = \frac{1}{4} \left\{ \overline{\langle W, (u \cdot v)Z \rangle}, \langle W, zW \rangle \right\} \\
&= \frac{1}{4} \langle W, (u \cdot v \cdot z)W \rangle = \frac{1}{4} \langle W, (z \cdot v \cdot u)W \rangle \\
&= \frac{1}{8} \langle W, (z \cdot v \cdot u + u \cdot v \cdot z)W \rangle \\
&= \mathcal{X}_{\{uvz\}}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\{\mathcal{S}_{uv}, \mathcal{Y}_z\} &= \frac{1}{2} \{ \langle W, (u \cdot v)Z \rangle, \langle Z, zZ \rangle \} = \left\{ \overline{\langle W, (u \cdot v)Z \rangle}, \langle Z, zZ \rangle \right\} \\
&= -\langle Z, (z \cdot u \cdot v)Z \rangle = -\langle Z, (v \cdot u \cdot z)Z \rangle \\
&= -\frac{1}{2} \langle Z, (z \cdot u \cdot v + v \cdot u \cdot z)Z \rangle \\
&= -\mathcal{Y}_{\{vuz\}}.
\end{aligned}$$

Since functions  $\mathcal{S}_{uv}$ ,  $\mathcal{X}_z$  and  $\mathcal{Y}_z$  on  $T\mathbb{H}_*^n$  (actually on  $T^*\mathbb{H}_*^n$ ) are  $\text{Sp}(1)$ -invariant, and the action of  $\text{Sp}(1)$  on  $T^*\mathbb{H}_*^n$  is symplectic, we have a Poisson realization on the Poisson manifold  $T^*\mathbb{H}_*^n/\text{Sp}(1)$  for the conformal algebra of the Jordan algebra  $\mathbb{H}_n(\mathbb{H})$ .

(iii). Since  $\mathcal{L}_u = \frac{1}{2}\langle W, uZ \rangle$ , we have

$$\begin{aligned}
\frac{2}{n} \sum_{\alpha} \mathcal{L}_{e_{\alpha}}^2 &= \frac{1}{2n} \sum_{\alpha} \langle W, e_{\alpha}Z \rangle^2 = \frac{n}{2} \sum_{\alpha} \langle (ZW^\dagger)_+ | e_{\alpha} \rangle^2 \\
&= \frac{n}{2} \langle (ZW^\dagger)_+ | (ZW^\dagger)_+ \rangle = \frac{1}{2} \text{tr} ((ZW^\dagger)_+)^2 \\
&= \frac{1}{8} \text{tr} (ZW^\dagger + WZ^\dagger)^2 \\
&= \frac{1}{8} \text{tr} (Z(W^\dagger ZW^\dagger) + (WZ^\dagger W)Z^\dagger + |W|^2 ZZ^\dagger + |Z|^2 WW^\dagger) \\
&= \frac{1}{8} ((W^\dagger Z)^2 + (Z^\dagger W)^2) + \frac{1}{4} |W|^2 |Z|^2 \\
&= \frac{1}{16} (W^\dagger Z - Z^\dagger W)^2 + \frac{1}{16} (W^\dagger Z + Z^\dagger W)^2 + \frac{1}{4} |W|^2 |Z|^2 \\
&= -\frac{1}{4} |\text{Im}(W^\dagger Z)|^2 + \frac{1}{4} \langle W, Z \rangle^2 + \frac{1}{4} |W|^2 |Z|^2 \\
&= -\mu^2 + \mathcal{L}_e^2 + \mathcal{X}_e \mathcal{Y}_e.
\end{aligned}$$

□

**Proof of Theorem 1.** As we remarked early that the Poisson realization is obviously unique, if it exists. From part (ii) of the above lemma, we know that there is a Poisson realization for  $\mathfrak{so}^*(4n)$  on the Poisson manifold  $T^*\mathbb{H}_*^n/\mathrm{Sp}(1)$ , hence on each of its symplectic leave. Since a Sternberg phase space is symplectic equivalent to a symplectic leave of  $T^*\mathbb{H}_*^n/\mathrm{Sp}(1)$ , in view of part (i) of the above lemma, one obtains the first part of Theorem 1. The remaining part of Theorem 1 follows trivially from part (iii) of the above lemma.

**Remark 4.1.** *The Poisson realization in Theorem 1 is called the dynamical symmetry for the  $\mathrm{Sp}(1)$ -Kepler problem at level  $n$  and with magnetic charge  $\mu$ .*

**Remark 4.2.** *The natural identification of Sternberg phase spaces with symplectic leaves of  $T^*\mathbb{H}_*^n/\mathrm{Sp}(1)$  is a bijection, so the totality of Sternberg phase spaces is naturally equivalent to the Poisson manifold  $T^*\mathbb{H}_*^n/\mathrm{Sp}(1)$ . This really indicates that  $\mathrm{Sp}(1)$  Kepler problems can be obtained via symplectic-reduction from a model whose phase space is  $T^*\mathbb{H}_*^n$  and whose hamiltonian is  $\mathrm{Sp}(1)$ -invariant, namely the  $n$ th quaternionic conformal Kepler problem in section 6 of Ref. [12].*

**Remark 4.3.** *In view of Ref. [9], Theorem 1 implies that the corresponding  $\mathrm{Sp}(1)$ -Kepler problem is the Hamiltonian system with phase space  $TC_1$ , Hamiltonian*

$$H = \frac{1}{2} \frac{\mathcal{X}_e}{\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e}$$

and Laplace-Runge-Lenz vector

$$\mathcal{A}_u = \frac{1}{2} \left( \mathcal{X}_u - \mathcal{Y}_u \frac{\mathcal{X}_e}{\mathcal{Y}_e} \right) + \frac{\mathcal{Y}_u}{\mathcal{Y}_e}$$

where  $\mathcal{X}_u$  and  $\mathcal{Y}_u$  are the functions that represent  $X_u$  and  $Y_u$  respectively in the Poisson realization in Theorem 1. Indeed, a simple computation yields

$$H = \frac{1}{2} \frac{\langle x | \pi^2 \rangle}{r} + \frac{\mu^2}{2r^2} - \frac{1}{r},$$

i.e., the Hamiltonian in Definition 1.1 of Ref. [12]. Here  $r = \frac{\mathrm{tr} x}{n}$ .

## 5. QUADRATIC RELATIONS AND ENERGY FORMULAE

For the Poisson realization in Theorem 1, let us assume that the elements of the conformal algebra such as  $S_{uv}$ ,  $X_z$  and  $Y_w$  are realized as functions

$$\mathcal{S}_{u,v}, \quad \mathcal{X}_z, \quad \mathcal{Y}_w$$

respectively. We shall use  $\mathcal{L}_u$  to denote  $\mathcal{S}_{ue}$  and  $\mathcal{L}_{u,v}$  to denote  $\frac{1}{2}(\mathcal{S}_{uv} + \mathcal{S}_{vu})$ .

The main purpose of this section is to list two corollaries of Theorem 1, one concerning the secondary quadratic relations and one concerning a formula connecting the Hamiltonian to the angular momentum and the Laplace-Runge-Lenz vector. The proof can be taken verbatim from the last section of Ref. [10].

**Corollary 1.** *Let  $e_\alpha$  be an orthonormal basis for  $\mathbb{H}_n(\mathbb{H})$ . In the following we hide the summation sign over  $\alpha$  or  $\beta$ . For the Poisson realization in Theorem 1, we have the following secondary quadratic relations:*

$$(i) \quad \mathcal{X}_{e_\alpha} \mathcal{L}_{e_\alpha} = n \mathcal{X}_e \mathcal{L}_e, \quad \mathcal{Y}_{e_\alpha} \mathcal{L}_{e_\alpha} = n \mathcal{Y}_e \mathcal{L}_e,$$

$$(ii) \quad \frac{1}{n} \mathcal{L}_{e_\alpha, u} \mathcal{L}_{e_\alpha} = -\mathcal{X}_u \mathcal{Y}_e + \mathcal{X}_e \mathcal{Y}_u,$$

$$(iii) \mathcal{X}_{e_\alpha}^2 = n\mathcal{X}_e^2, \mathcal{Y}_{e_\alpha}^2 = n\mathcal{Y}_e^2,$$

$$(iv) \frac{2}{n}\mathcal{L}_{e_\alpha, u}\mathcal{X}_{e_\alpha} = -\mathcal{X}_u\mathcal{L}_e + \mathcal{L}_u\mathcal{X}_e, \frac{2}{n}\mathcal{L}_{e_\alpha, u}\mathcal{Y}_{e_\alpha} = \mathcal{Y}_u\mathcal{L}_e - \mathcal{L}_u\mathcal{Y}_e,$$

$$(v) \mathcal{X}_{e_\alpha}\mathcal{Y}_{e_\alpha} = n(\mathcal{L}_e^2 + \mu^2),$$

$$(vi) \frac{4}{n^3}\mathcal{L}_{e_\alpha, e_\beta}^2 = \mathcal{X}_e\mathcal{Y}_e - \mathcal{L}_e^2 + \frac{n-2}{n}\mu^2.$$

**Corollary 2.** *Let  $e_\alpha$  be an orthonormal basis for  $H_n(\mathbb{H})$ ,  $L^2 = \frac{1}{2}\sum_{\alpha, \beta}\mathcal{L}_{e_\alpha, e_\beta}^2$ , and  $A^2 = -1 + \sum_{\alpha}\mathcal{A}_{e_\alpha}^2$ . Then the Hamiltonian  $H$  satisfies the relation*

$$(5.1) \quad -2H\left(L^2 - \frac{n^2(n-1)}{4}\mu^2\right) = \left(\frac{n}{2}\right)^2(n-1-A^2).$$

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